1. Proofs

**Proposition 1.** For an initial resource level \( r \in \mathcal{R} \), a vector of intra-hour prices \( P \in \mathbb{R}^M \), a bid \( b = (b^-, b^+) \in \mathcal{B} \), and a subinterval \( m \), the resource transition function \( g^R_m(r, q(P, b)) \) is nondecreasing in \( r \), \( b^- \), and \( b^+ \).

**Proof.** Since
\[
g^R_{m+1}(R_t, q_s) = \min\{g^R_m(R_t, q_s) - e_m^T q_s, R_{\max}\},
\]

it is clear that the transition from \( g^R_m \) to \( g^R_{m+1} \) is nondecreasing in the value of \( g_m \) and nonincreasing in the value of \( e_m q_s \). Thus, a simple induction argument shows that for \( r_1, r_2 \in \mathcal{R} \) and \( q_1, q_2 \in \{-1, 0, 1\}^M \) where \( r_1 \leq r_2 \) and \( q_1 \leq q_2 \),
\[
g^R_M(r_1, q_2) \leq g^R_M(r_2, q_1).
\]

The result follows from the fact that \( q(P, b) \) is nonincreasing in \( b \).

**Proposition 2.** For an initial \( l \in \mathcal{L} \), a vector of intra-hour prices \( P \in \mathbb{R}^M \), a bid \( b = (b^-, b^+) \in \mathcal{B} \), and a subinterval \( m \), the transition function \( g^L_m(l, d(P, b)) \) is nondecreasing in \( l \), \( b^- \), and \( b^+ \).

**Proof.** The transition
\[
g^L_{m+1}(L_t, d_s) = [g^L_m(L_t, d_s) - e_m^T d_s]^+
\]
is nondecreasing in \( g^L_m \) and nonincreasing in \( e_m d_s \). Like in Proposition 1, induction shows that for \( l_1, l_2 \in \mathcal{L} \) and \( d_1, d_2 \in \{0, 1\}^M \) where \( l_1 \leq l_2 \) and \( d_1 \leq d_2 \),
\[
g^L_M(l_1, d_2) \leq g^L_M(l_2, d_1).
\]
The result follows from the fact that \( d(P, b) \) is nonincreasing in \( b \).

**Proposition 3.** The contribution function \( C_{t,t+2}(S_t, b_t) \), with \( S_t = (R_t, L_t, b_{t-1}, P^S_t) \) is nondecreasing in \( R_t, L_t, b_{t-1}^- \), and \( b_{t-1}^+ \).

**Proof.** First, we argue that the revenue function \( C(r, l, P, b) \) is nondecreasing in \( r \) and \( l \). From their respective definitions, we can see that \( \gamma_m \) and \( U_m \) are both nondecreasing in their first arguments. These arguments can be written in terms of \( r \) and \( l \) through the transition functions \( g^R_m \) and \( g^L_m \). Applying Proposition 1 and Proposition 2, we can confirm that \( C(r, l, P, b) \) is nondecreasing in \( r \) and \( l \). By its definition,
\[
C_{t,t+2}(S_t, b_t) = \mathbb{E}\left[ C\left( g^R(R_t, P_{(t,t+1)}, b_{t-1}^-), g^L(L_t, P_{(t,t+1)}, b_{t-1}^-), P_{(t+1,t+2)}, b_t\right) | S_t \right].
\]

Again, applying Proposition 1 and Proposition 2 (for \( m = M \), we see that the term inside the expectation is nondecreasing in \( R_t, b_{t-1}^- \), and \( b_{t-1}^+ \) (composition of nondecreasing functions) for any outcome of \( P_{(t,t+1)} \) and \( P_{(t+1,t+2)} \). Thus, the expectation itself is nondecreasing.

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Lemma 1. Define deterministic bounding sequences \( L^k_t \) and \( U^k_t \) in the following way. Let \( U^0 = V^* + V_{\max}, e \) and \( L^0 = V^* - V_{\max}, e \), where \( e \) is a vector of ones. In addition, \( U^{k+1} = (U^k + HU^k)/2 \) and \( L^{k+1} = (L^k + HL^k)/2 \). Then, for each \( s \in S^b \) and \( t \leq T - 1 \),

\[
L^k_t(s) \rightarrow V^b_t(s),
\]

\[
U^k_t(s) \rightarrow V^b_t(s),
\]

where the limit is in \( k \).

Proof. We first show that \( H \) satisfies the following properties:
Lemma 2. \(U^k\) and \(L^k\) both satisfy the monotonicity property: for each \(t, k\), and \(s_1, s_2 \in S^b\) such that \(s_1 \preceq_b s_2\),

\[
\begin{align*}
U^k_t(s_1) & \leq U^k_t(s_2), \\
L^k_t(s_1) & \leq L^k_t(s_2).
\end{align*}
\]

Proof. To show this, first note that given a fixed \(t \leq T - 2\) and any vector \(Y \in \mathbb{R}^{[S^b]}\) (defined over the post-decision state space) that satisfies the monotonicity property, it is true that the vector \(h_t Y\), whose component at \(s \in S^b\) is defined using the post-decision Bellman recursion,

\[
(h_t Y)(s) = \mathbb{E}\left[ \max_{b_{t+1} \in B} \left\{ C_{t+1, t+3}(S_{t+1}, b_{t+1}) + Y(S^b_{t+1}) \right\} | S^b_t = s \right],
\]

also obeys the monotonicity property. We point out that there is a small difference between the operator \(H\) and \(h_t\) in that \(H\) operates on vectors of dimension \(T \cdot |S^b|\). To verify monotonicity, \(s_1, s_2 \in S^b\) such that \(s_1 \preceq_b s_2\). For a fixed sample path of prices \(P\), let \(S_{t+1}(s_1, P)\) and \(S_{t+1}(s_2, P)\) be the respective downstream pre-decision states. Applying Propositions 1 and 2, we have that \(S_{t+1}(s_1, P) \preceq_b S_{t+1}(s_2, P)\). For any fixed \(b_{t+1} \in B\), we apply the monotonicity of the contribution function \(C_{t+1, t+3}\) (Proposition 3) and the monotonicity of \(Y\) to see that

\[
C_{t+1, t+3}(S_{t+1}(s_1, P), b_{t+1}) + Y((S_{t+1}(s_1, P), b_{t+1})) \leq C_{t+1, t+3}(S_{t+1}(s_2, P), b_{t+1}) + Y((S_{t+1}(s_2, P), b_{t+1})),
\]

which confirms that \((h_t Y)(s_1) \leq (h_t Y)(s_2)\). When \(t = T - 1\), we set \((h_t Y)(s) = \mathbb{E}[C_{t+1}(S_{t+1}) | S^b_t = s]\) and the same monotonicity result holds.

Now, we can easily proceed by induction on \(k\), noting that \(U^0\) and \(L^0\) satisfy monotonicity for each \(t\). Assuming that \(U^k\) satisfies monotonicity, we can argue that \(U^{k+1}\) does as well; we first note that for any \(t\), by the definition of \(U^{k+1}\),

\[
U^{k+1}_t = \frac{U^k_t + (H U^k)_t}{2} = \frac{U^k_t + (h_t U^k + h_t U^k + h_t U^k + h_t U^k)}{2}.
\]

By the induction hypothesis and the property of \(h_t\) proved above, it is clear that \(U^{k+1}\) also satisfies monotonicity and the proof is complete. \(\square\)