

# Outline



- Notational style

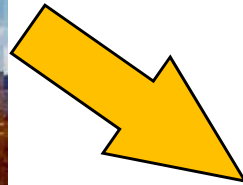
- The budgeting problem

  - » The basic budget problem

  - » Budgeting with an S-curve effect

  - » The dynamic budgeting problem

# Notational style

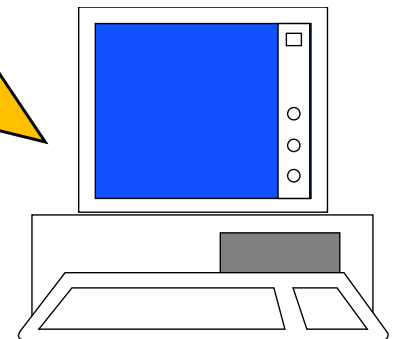
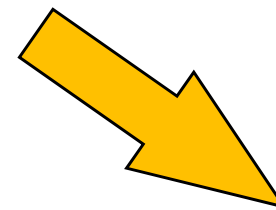


*Our challenge is converting real problems into a form that can be studied on the computer.*

$$\min_{\pi} \mathbb{E} \sum_{t=0}^T C(S_t, X_t^{\pi}(S_t))$$

$$x_t = X_t^{\pi}(S_t) \in \mathcal{X}_t$$

$$S_{t+1} = S^M(S_t, x_t, W_{t+1})$$



# Notational style

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## ■ The challenge of communicating:

- » “The elevator lift is being fixed for the next day. During that time we regret that you will be unbearable.”  
- Sign in a Bucharest business.
- » “We will take your bags and send them in all directions.” - Airline office in Copenhagen.
- » “You are invited to take advantage of the chambermaid.” - A Japanese hotel.
- » “Please leave your values at the front desk” - sign at a Paris hotel
- » From a tailor shop: “Place orders early because in a big rush we will execute customers in strict rotation.”

# Notational style

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## ■ The challenge of communicating:

- » In an Acapulco hotel: “The manager has personally passed all the water served here.”
- » “Please do not feed the animals. If you have any suitable food, give it to the guard on duty.” - Sign at a Budapest zoo.
- » In a Swiss eatery: “Our wines leave you nothing to hope for.”
- » “Ladies, leave your clothes here and spend the afternoon having a good time.” A laundry in Rome.

# Notational style

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## ■ Principles of good mathematical notation:

- » Notation is a language - if the language is hard to learn, others will have difficulty speaking it.
- » Minimize the number of variables you introduce (i.e. keep your vocabulary small).
- » Make variables as mnemonic as possible.
- » Follow consistent, standard conventions.
- » Organize your variables into natural groupings:

# Notational style

## ■ Variables

- » A basic variable - lower case and script:  $x$
- » Use subscripts to identify elements of a vector:

$$x_{tij} \quad x_{ti} \quad x_t$$

- » Use superscripts to create different flavors of a variable:

**Decisions :**

**Costs :**

**purchase:**  $x^p$

$c^p$

**sell :**  $x^s$

$c^s$

**hold :**  $x^h$

$c^h$

$$x = \{x^p, x^s, x^h\} \quad c = \{c^p, c^s, c^h\}$$

$$\text{Total costs} = c^p x^p + c^s x^s + c^h x^h = cx$$

# Notational style

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## ■ Variables

» *Never* use variables with more than one letter:

$PC$  = Purchasing cost (is it  $P$  times  $C$ ?)

$BC_t$  = Battery charge at time  $t$

» Using multiple letter variables is popular in the business community. If you are presenting an equation in a memo to a management audience, it is perfectly reasonable to write:

$$Fleet_t = Const_0 + Const_1 \times Tons_t + Const_2 \times speed_t$$

This is acceptable in this setting, because the variables define themselves, and you are never going to manipulate this equation.

# Notational style

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## ■ Sequencing subscripts:

» When you have multiple subscripts:

- Roughly sequence them in the order that you would place them in a summation:

$x_{ti}$  = Dollars invested in asset  $i$  in time period  $t$ .

$$C(x) = \sum_t \sum_i c_{ti} x_{ti}$$

$$x_t = (x_{ti})_{i \in \mathcal{I}}$$

- Always model discrete time periods as a subscript, because it is an element of a vector over different time periods.



# Notational style

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## ■ Arguments, superscripts and hats:

- » Use arguments only when there is a functional dependence:

$$F(x) \quad \text{or} \quad a(x)$$

- » If we are estimating a variable iteratively, again use a superscript to identify different versions of the variable:

$$x^{n+1} = F(x^n)$$

Use  $n$  for iterations. If you need outer and inner iterations, use  $n$  and  $m$ .

- » Batch vs. recursive data

In statistics, it is common to represent a set of  $n$  data points

$(x_1, x_2, \dots, x_n, \dots, x_N)$ , where  $x_n$  is a vector with element  $x_n^i$ .

# Notational style

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## ■ Choice of letters:

- » Use hats and bars (etc) to indicate different estimates/observations of the same variable:

$$\bar{x}, \hat{x}$$

- » Constants:

- General rule:  $a, b, c, d$  and  $e$

- » Physical parameters (ratios, speeds, ...):

- Use Greek letters:

$$\alpha \quad \beta \quad \gamma \quad \lambda \quad \rho \quad \delta$$

- Avoid unusual Greek letters, e.g.

$$\xi \quad \zeta \quad (\text{hard to pronounce/remember})$$

# Notational style

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## ■ Time:

- » Always use  $t$  or its variants.
  - A flight going from city  $i$  to city  $j$  might start at time  $t$  and arrive a time  $t'$ .
  - You might purchase an option at time  $t$  which can be exercised at time  $t'$ .
  
- » Use  $\tau$  to indicate an interval of time, such as the time required to complete an action.
  
- » Let  $t = 0$  represent “here and now.”

# Notational style

## ■ Modeling new information:

- » It helps to identify variables that represent new information arriving in a time period.
- » Suggest using “hats” to indicate new information:

$\hat{D}_t$  = Customer demand for a product in time  $t$ .

$\hat{p}_t$  = Market price of a stock at time  $t$ .

$\hat{b}_t$  = A baseball player's batting average in year  $t$ .

- » Use “bars” for statistics calculated from exogenous information:

$$\bar{D}_t = \frac{1}{t} \sum_{t'=1}^t \hat{D}_{t'}$$

$$\bar{p}_t = (1 - \alpha) \bar{p}_{t-1} + \alpha \hat{p}_t$$

$$\bar{b}^N = \frac{1}{N} \sum_{n=1}^N \hat{b}^n$$

# Notational style

## ■ Sets:

» Sets - Calligraphic (or script) capital letters

$x \in \mathcal{X}$        $a \notin \mathcal{A}$       “Monotype Corsiva” in equation editor

- Subsets - suggest using:

$\mathcal{X}_i$        $\mathcal{A}_b$

Calligraphic in Latex

» Use sets for summations:

- Sets are especially useful when there is not a natural indexing:

$\sum_{i \in \mathcal{I}} x_i$  is better than:  $\sum_{i=1}^n x_i$

Let  $a$  be a multiattribute vector, where:

$$a \in \mathcal{A}$$

It is then easy to write:

$$\sum_{a \in \mathcal{A}} x_a$$

- We can also write vectors:

$$x = (x_t)_{t \in \mathcal{T}} \quad x_t = (x_{ti})_{i \in \mathcal{I}}$$

# Notational style

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## ■ Notational style

» Functions:

Argument notation (preferred in engineering):

$$F(x)$$

$$F(x, y)$$

Mapping notation (preferred in mathematics):

$$F : \mathcal{X} \rightarrow \mathfrak{R}$$

$$F : \mathcal{X} \times \mathcal{Y} \rightarrow \mathfrak{R}$$

» Use lower case letters for individual functions; use upper case for sums:

$$C(x) = \sum_{t \in T} c_t(x_t)$$

# Notational style

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## ■ Notational style

### » Matrices:

- Square, capital letters:

$A, B$

### » Multiplication of matrices and vectors:

Use  $cx$  when it is understood that we want an inner product. The "transpose" is implicit. There is no need to write:

$$c^T x$$

### » When typing equations in MS Word...

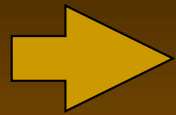
- I recommend purchasing the MathType add-in. Much better than Word's built in equation editor:

<http://www.dessci.com/en/products/mathtype/>

# Outline

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- » The basic budgeting problem

- » Budgeting with an S-curve effect

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# The budgeting problem

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## ■ The budgeting problem

- » How much of a single resource should be allocated to different activities (sometimes over time).
- » This can be viewed as taking a single “budget” and then setting the budgets of a series of smaller tasks.
- » Resources:
  - Money
    - Your division budget is fixed at the beginning of the year. You have to decide how your budget should be divided among your subdivisions.
  - Investment
    - You have to allocate your funds over different investments.
  - Time
    - How much time should you allocate to different assignments before Monday?
    - How much time should be given to individual tasks?

# *In Discarding of Kidneys, System Reveals Its Flaws*

## *1 in 5 Organs for Transplant Go Unused, and Experts Point to Rules for Allocation*

By KEVIN SACK

ST. PAUL — Last year, 4,720 people died while waiting for kidney transplants in the United States. And yet, as in each of the last five years, more than 2,600 kidneys were recovered from deceased donors and then discarded without being transplanted, government data show.

Those organs typically wound up in a research laboratory or medical waste incinerator.

In many instances, organs that seemed promising for transplant based on the age and health of the donor were discovered to have problems that made them not viable.

But many experts agree that a significant number of discarded kidneys — perhaps even half, some believe — could be transplanted if the system for allocating them better matched the right organ to the right recipient in the right amount of time.

considered simple and transparent. But many in the field argue that it wastes precious opportunities for transplants.

One recent computer simulation, by researchers with the Scientific Registry of Transplant Recipients, projected that a redesigned system could add 10,000 years of life from just one year of transplants.

Currently, the country is divided into 58 donation districts. When a deceased donor kidney becomes available, the transplant network's rules dictate that it is first offered to the compatible candidate within the district who has waited the longest. Additional priority is given to children, to candidates whose blood chemistry makes them particularly difficult to match and to those who are particularly well matched to the donor. If no taker is found locally, the electronic search ex-

# The budgeting problem

## ■ Basic model:

$R$  = Amount of a resource to be allocated over different activities  
(this is our "budget")

$\mathcal{T}$  = A set of tasks (or projects, or time periods) to which the resource  
should be allocated.

$= (1, 2, \dots, T)$

$x_t$  = Amount of the resource that should be allocated to task  $t$ .

$x = (x_1, x_2, \dots, x_T)$

= An *allocation*.

$C(x)$  = Total contribution from an allocation  $x$ .

# The budgeting problem

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To start, we are going to assume that  $C(x)$  is *separable*.

That is, we can write:

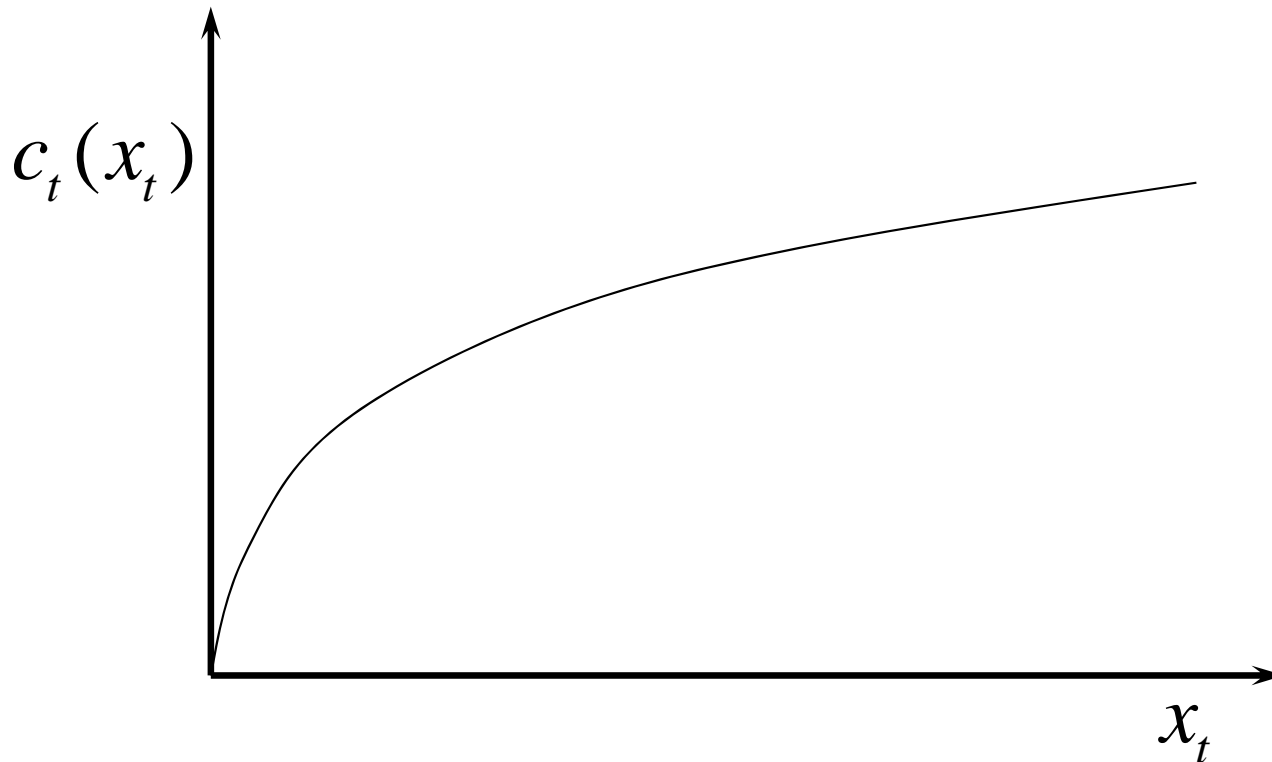
$$C(x) = \sum_{t \in T} c_t(x_t)$$

This means that the marginal value of increasing the allocation for task  $t$  is independent of the rest of the allocation vector.

# The budgeting problem

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The contribution function is typically concave (when we are maximizing):



# The budgeting problem

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## ■ “The” resource allocation problem

We wish to solve:

$$\max_x \sum_{t \in \mathcal{T}} c_t(x_t)$$

subject to:

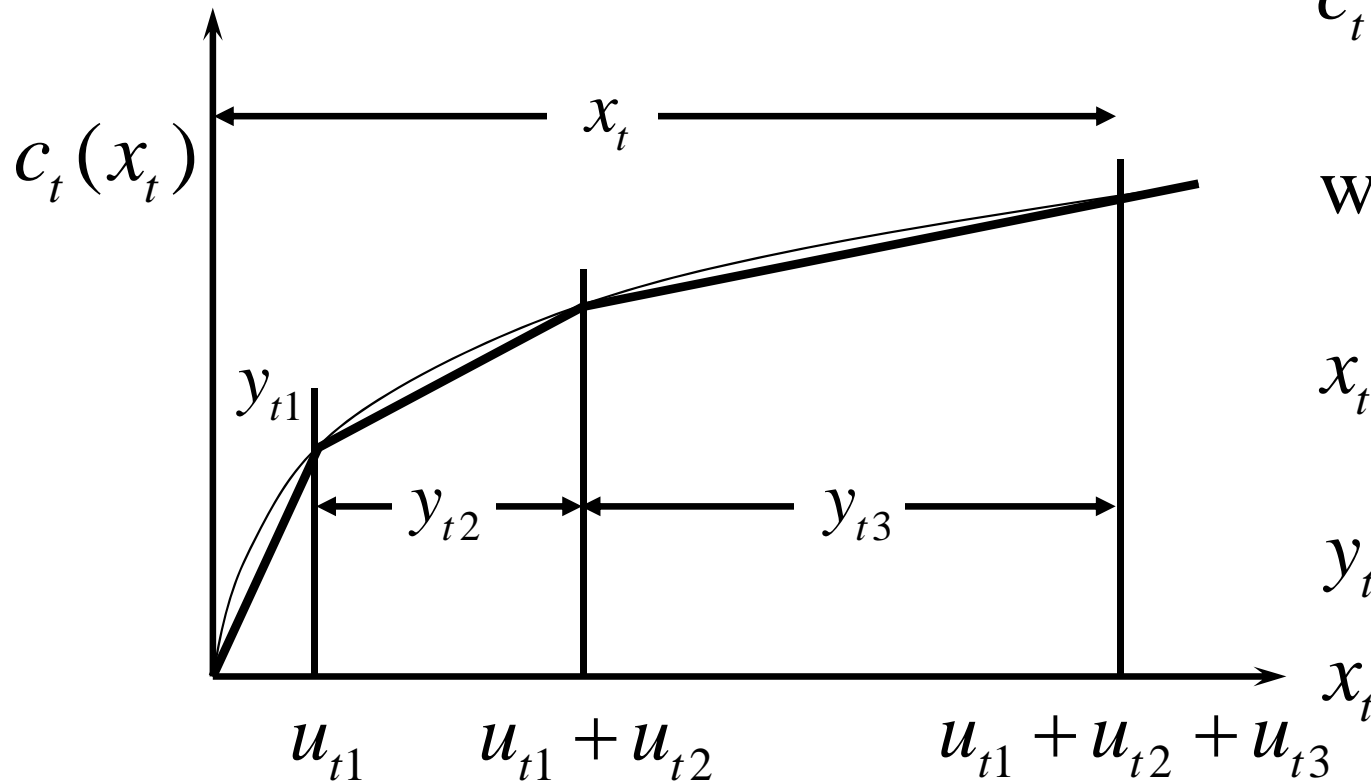
$$\sum_{t \in \mathcal{T}} x_t \leq R$$

"The resource constraint"

$$x_t \geq 0$$

# The budgeting problem

It is sometimes convenient to turn the contribution function into a piecewise linear approximation:



$$c_t(x_t) \approx \sum_{i \in \mathcal{I}_t} c_{ti} y_{ti}$$

where

$$x_t = \sum_i y_{ti},$$

$$y_{ti} \leq u_{ti}$$

# The budgeting problem

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## ■ Solving the budget allocation problem:

We can convert our original nonlinear program into a linear program using a standard modification:

$$\max_x \sum_{t \in T} \sum_{i \in \mathcal{I}_t} c_{ti} y_{ti}$$

subject to:

$$\sum_{t \in T} \sum_{i \in \mathcal{I}_t} y_{ti} \leq R$$

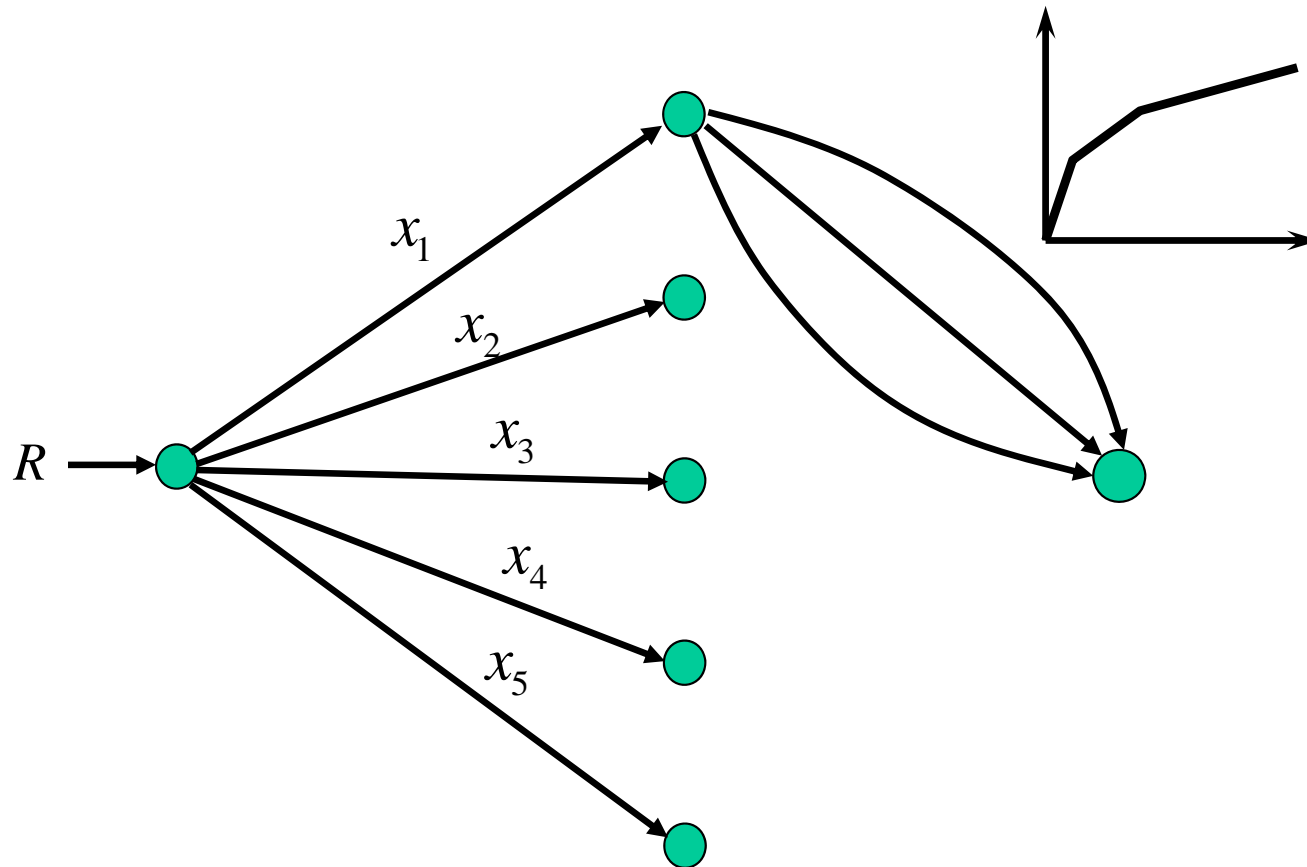
$$y_{ti} \leq u_{ti}$$

$$y_{ti} \geq 0$$



# The budgeting problem

- The budget allocation problem as a (linear) network:



# The budgeting problem

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## ■ Variations:

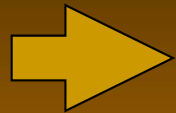
- » Contribution function may be nonconcave:
  - Volume discounts
  - Fixed cost for investing in a project
    - Time required to investigate the project
    - Fixed costs for allocating resources to the project
- » Contribution function may be nonseparable
  - You are drilling for oil. Different oil wells tap the same basic oil field. The output of one well reflects all the wells that are drilled (and their location).
- » Contribution may depend on the history of investments
  - Expenditures in HIV treatment; impact in time  $t$  depends on pattern of investment over time.
- » Evolving information
  - We may learn new information as the project unfolds

# Outline

- Notational style

- The budgeting problem

  - » The basic budgeting problem



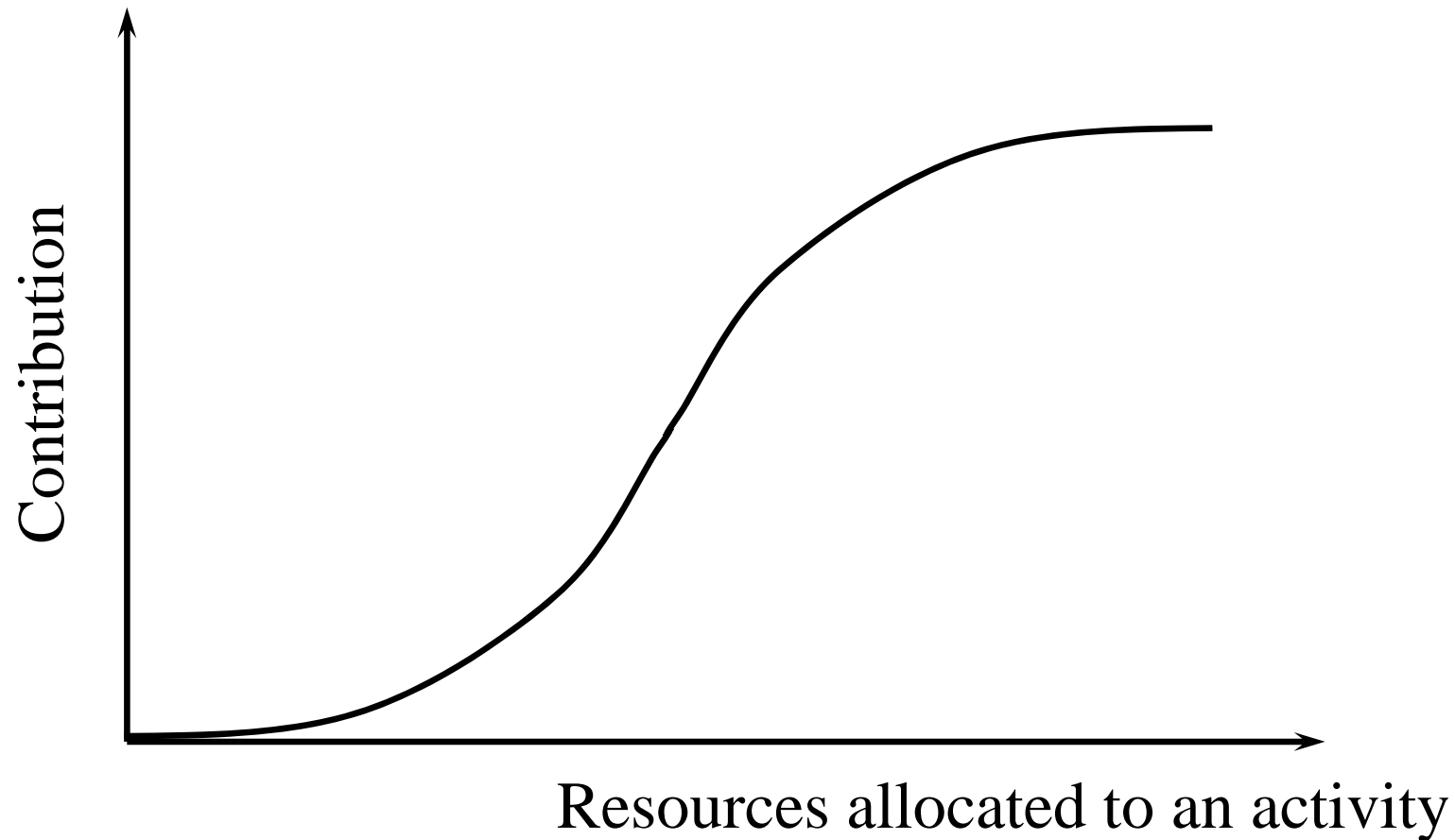


  - » Budgeting with an S-curve effect

  - » The dynamic budgeting problem

# The S-curve

- Resource allocation with an S-curve contribution function



# The S-curve

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## ■ Examples:

### » Airlines

- How many flying hours of their aircraft fleet should be dedicated to different markets?

### » Universities

- What areas of research should a university invest faculty resources in?

### » Auto manufacturers:

- What product segments should a company focus in?
- What geographical regions should the company focus on selling to?

# The S-curve

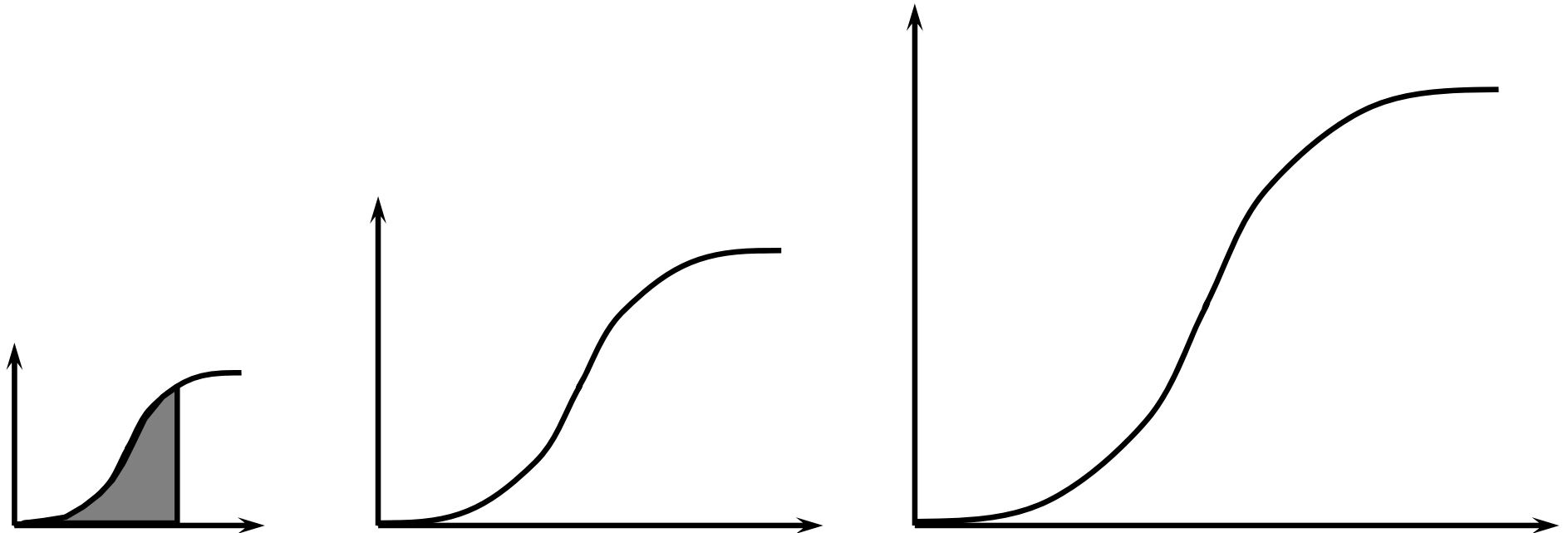
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- Example: airlines
  - » Mixture of markets of different sizes
- What would you do if you were a small airline?
  
- Medium sized?
  
  
- Large?

# The S-curve

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- If you have limited resources:
  - » Saturate the smallest markets.

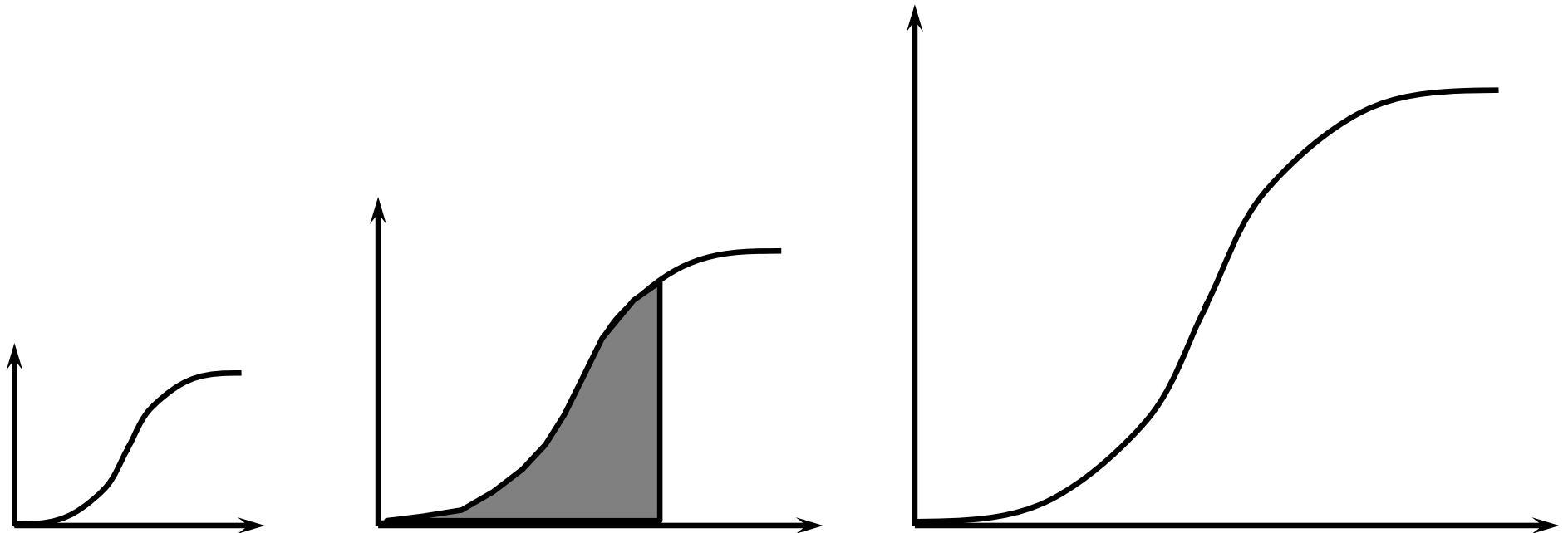


# The S-curve

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## ■ If you are medium-sized:

- » Put everything you have in the biggest markets that you can dominate, whether they be the mid-sized...

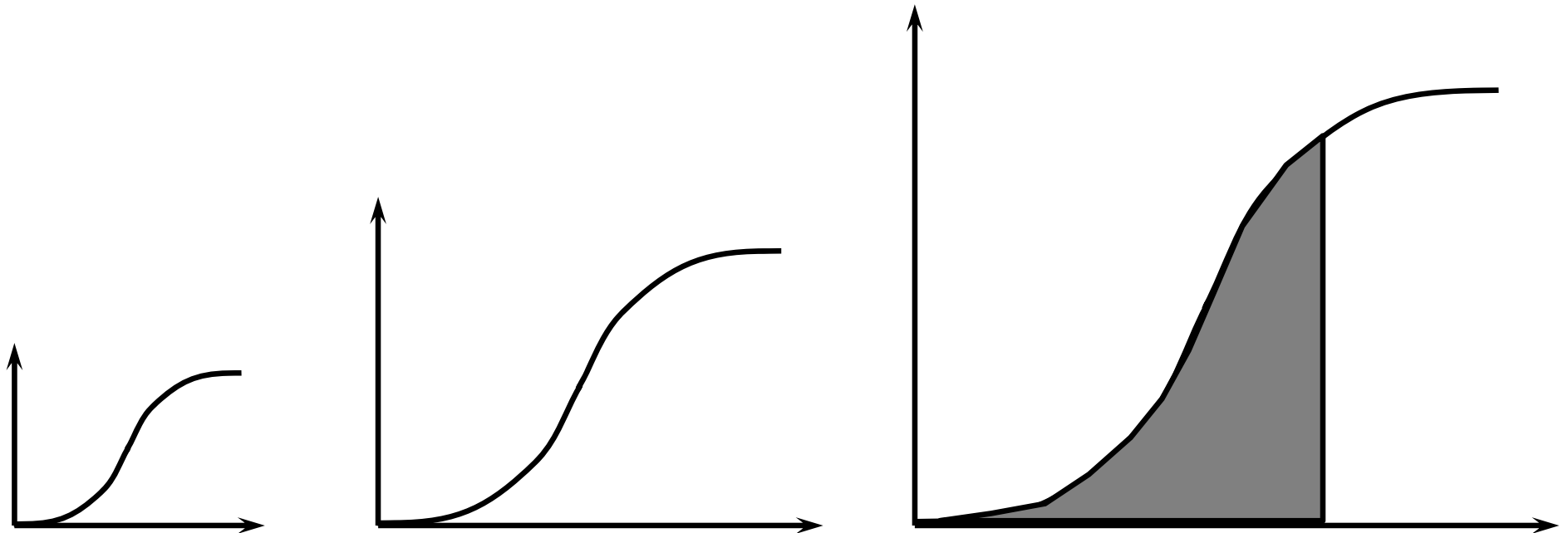




# The S-curve

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- If you are medium-sized:
  - » ... or the biggest markets.

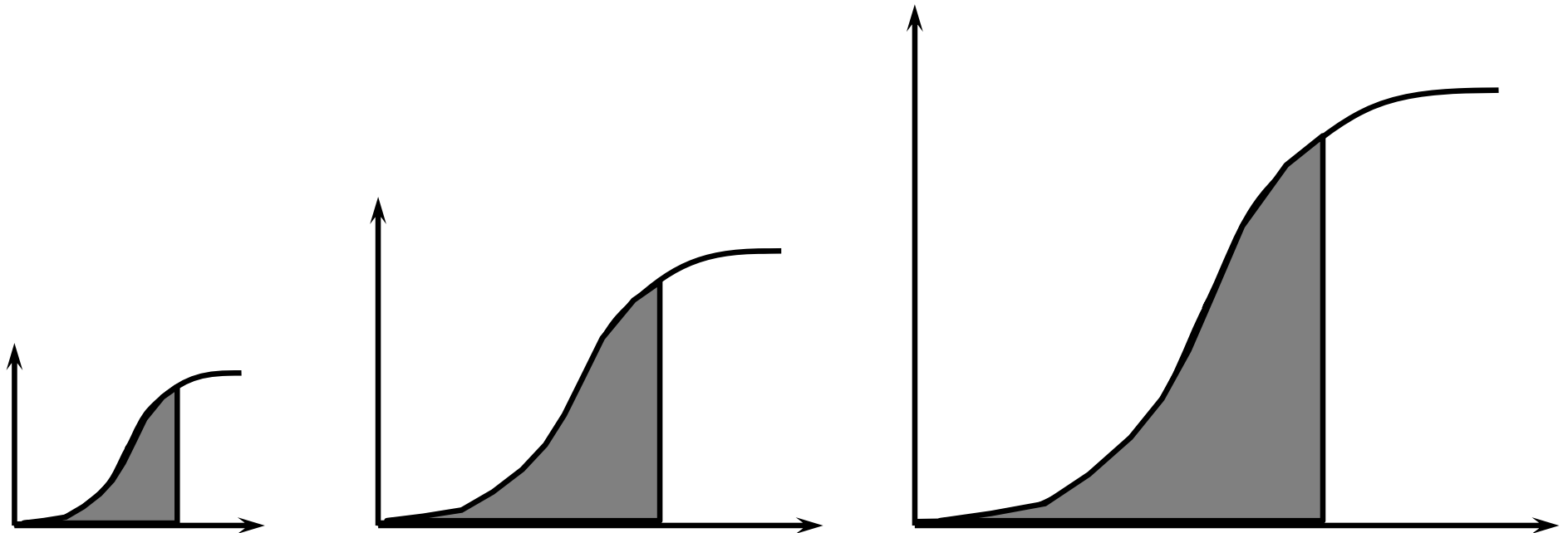


# The S-curve

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- If you are big enough:

- » Allocate your resources across all markets in proportion to their size:



# The S-curve

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## ■ The Canadian coffee market

In November 2000, Starbucks, the world's leading gourmet coffee retailer, announced that it would open around 50 to 75 coffee shops in the province of Quebec in the following four years. At the time, it had only five outlets compared with the 38 stores of Second Cup, the Canadian leader in the industry.

Perhaps foreseeing the threat, Second Cup had earlier abandoned its expansion plans in the United States and decided to concentrate its efforts on its Canadian business. Within 15 months of the announcement, it opened 18 new shops, increasing its number to 56. During the same period, Starbucks was able to open only three more shops and later opened another four, for a total of 12 by December 2003.

If the expansion observed in those three years is any indication, it is fair to conclude that Starbucks will obtain less than a quarter of the market it had hoped to capture initially. Although there may be other reasons why Starbucks' plan did not materialize, it is quite plausible that Second Cup's expansion played a role.

*Dasci and Laporte, "A Continuous Model for Multistore Competitive Location," Operations Research, Vol 53, 2005.*

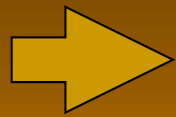
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  - » The basic budgeting problem

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  - » The dynamic budgeting problem

# The dynamic budgeting problem

Assume that  $\mathcal{T}$  represents a set of time periods, so that the budget allocation decisions must be made sequentially:  $x_1$  before  $x_2$  before  $x_3$ ... (we can retain the interpretation that  $\mathcal{T}$  is a set of tasks and then arbitrarily order them).

Now let:

$R_1$  = Our original budget (initial resource allocation)

$R_t$  = The amount of our original resource remaining before we make the decision for time  $t$ .

$V_t(R_t)$  = The value of having  $R_t$  remaining before solving time  $t$ .

We suggest that it is "intuitively obvious" that the following equation holds:

$$V_t(R_t) = \max_{x_t} \left\{ c_t(x_t) + V_{t+1}(R_{t+1}(R_t, x_t)) \right\}$$

# The dynamic budgeting problem

Often we will write:

$$V_t(R_t) = \max_{x_t} \{c_t(x_t) + V_{t+1}(R_{t+1})\}$$

where the functional dependence:

$$R_{t+1} = R_{t+1}(R_t, x_t) = R_t - x_t$$

is implicit.

We solve this using *backward dynamic programming*. Assume that  $\mathcal{T} = (1, 2, \dots, T)$ . We next have to assume that  $V_{T+1}(R_{T+1})$  is known (for example, we may simply assume that  $V_{T+1}(R_{T+1}) = 0$ ). We would then solve:

$$V_T(R_T) = \max_{x_T} \{c_T(x_T)\}$$

for all possible values of  $R_T$ . This is easiest if we consider only a discrete set of values for  $R_T$ .

# The dynamic budgeting problem

We can continue this process backward through time. Assume now that we know  $V_{t+1}(R_{t+1})$  for all possible values of  $R_{t+1}$ . For example, if we are deciding how much of our financial budget we can spend in month  $t$ , then we might restrict our attention to budgets in increments of \$10,000, and consider only expenditures in the same units. In this case, if we know  $V_{t+1}(R_{t+1})$  for values of  $R_{t+1} = (0, 10000, 20000, \dots)$ , then we can solve the equation:

$$V_t(R_t) = \max_{x_t} \{c_t(x_t) + V_{t+1}(R_t - x_t)\}$$

for values of  $R_t = (0, 10000, 20000, \dots)$ . For each value of  $R_t$ , we would evaluate the term  $(c_t(x_t) + V_{t+1}(R_t - x_t))$  for  $x_t = (0, 10000, 20000, \dots, R_t)$  (assuming we cannot spend more than what is in the budget - of course, we might be able to borrow money).

# The dynamic budgeting problem

## ■ Analytical example:

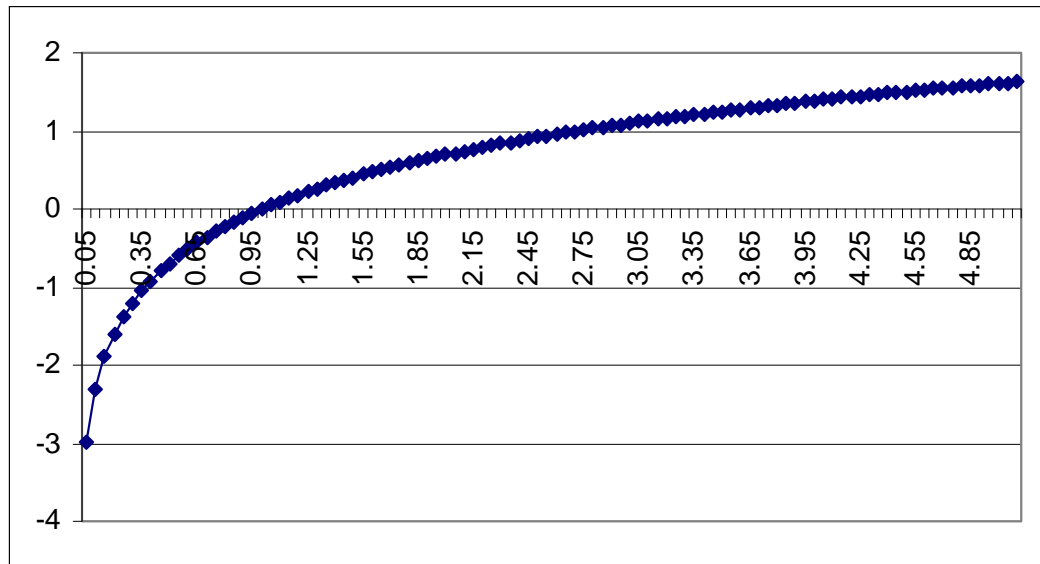
Assume that:

$$c_t(x_t) = c_t \ln x_t$$

This means:

$$\begin{aligned} V_T(R_T) &= \max_{x_T \leq R_T} c_T \ln x_T \\ &= c_T \ln R_T \end{aligned}$$

$\ln(x)$  function implies declining marginal returns for resources (which is common in practice).  
What do you think would be an optimal allocation with this type of function?





# The dynamic budgeting problem

## ■ Analytical example (cont'd):

$$\begin{aligned} V_{T-1}(R_{T-1}) &= \max_{x_{T-1} \leq R_{T-1}} \{c_{T-1} \ln x_{t-1} + V(R_{T-1} - x_{T-1})\} \\ &= \max_{x_{T-1} \leq R_{T-1}} \{c_{T-1} \ln x_{T-1} + c_T \ln(R_{T-1} - x_{T-1})\} \end{aligned} \quad f(x_{T-1}, R_{T-1})$$

differentiating:  $\frac{\partial f(x_{T-1}, R_{T-1})}{\partial x_{T-1}} = \frac{c_{T-1}}{x_{T-1}} - \frac{c_T}{R_{T-1} - x_{T-1}} = 0$

$$\Rightarrow c_{T-1}(R_{T-1} - x_{T-1}) = c_T x_{T-1}$$

$$\Rightarrow c_{T-1} R_{T-1} = (c_T + c_{T-1}) x_{T-1}$$

$$\Rightarrow x_{T-1} = \frac{c_{T-1} R_{T-1}}{c_T + c_{T-1}}$$

$$V_{T-1}(R_{T-1}) = c_{T-1} \ln \frac{c_{T-1} R_{T-1}}{c_T + c_{T-1}} + c_T \ln \left( R_{T-1} - \frac{c_{T-1} R_{T-1}}{c_T + c_{T-1}} \right)$$

$$= c_{T-1} \ln \frac{c_{T-1} R_{T-1}}{c_T + c_{T-1}} + c_T \ln \left( \frac{c_T R_{T-1}}{c_T + c_{T-1}} \right)$$

# The dynamic budgeting problem

## ■ Analytical example (cont'd):

Using "proof by extrapolation," let:

$$V_t(R_t) = \sum_{t'=t}^T c_{t'} \ln \gamma_{t,t'} R_t$$

where:

$$\gamma_{t,t'} = \frac{c_{t'}}{c_t + c_{t+1} + \dots + c_T}$$

This allows us to express:

$$x_t = \gamma_{t,t} R_t$$

*This is the fraction of the budget we expect to be allocated to time period  $t'$  when we are in time period  $t$ . Note that when we are in time period  $t$ , we don't actually do an allocation to time periods  $t' > t$ .*

Example: What if  $c_t = 1$  (all time periods are equally important)?

Then:

$$\gamma_{t,t} = \frac{1}{T - t + 1}$$

This means: allocate the budget equally over the remaining time periods.

# The dynamic budgeting problem

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## ■ Another example:

What solution would you expect if:

$$c_t(x) = c_t x^2$$